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Properties on relative normality, their absolute embeddings and related problems

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1. Introduction

This note is a summary of [20]. Throughout this paper all spaces are assumed to be T_1 topological spaces and the symbol γ denotes an infinite cardinal.

The notions of relative normality and relative paracompactness are central in the study of relative topological properties which has been posed by Arhangel'skiĭ and Genedi [4], and also in the subsequent articles [2] and [3] by Arhangel'skiĭ.

Let X be a space and Y a subspace of X. A subspace Y is said to be *normal* (respectively, *strongly normal*) *in* X if for each disjoint closed subsets F_0, F_1 of X (respectively, of Y), there exist disjoint open subsets G_0, G_1 of X such that $F_i \cap Y \subset G_i$ for each $i = 0, 1$. A subspace Y is said to be 1- (respectively, 2-) *paracompact in* X if for every open cover $\mathcal{U}$ of X, there exists a collection $\mathcal{V}$ of open subsets of X with $X = \bigcup \mathcal{V}$ (respectively, $Y \subset \bigcup \mathcal{V}$) such that $\mathcal{V}$ is a partial refinement of $\mathcal{U}$ and $\mathcal{V}$ is locally finite at each point of Y. Here, $\mathcal{V}$ is said to be a *partial refinement* of $\mathcal{U}$ if for each $V \in \mathcal{V}$, there exists a $U \in \mathcal{U}$ containing V. The term "2-paracompact" is often simply said "paracompact". In the definition of 2-paracompactness of Y in X above, when we replace "open cover of X" by "collection of open subsets of X with $Y \subset \bigcup \mathcal{U}$", Y is said to be *Aull-paracompact in* X ([3], [5]). Each of 1-paracompactness and Aull-paracompactness of Y in X clearly implies 2-paracompactness of Y in X. Note that 1-paracompactness coincides with α-paracompactness defined by Aull [6] for a closed subset of a regular space [23]. See also Theorem 3.11.

On the other hand, it is natural to define the following two relative notions; a subspace Y of a space X is said to be γ-*collectionwise normal* (respectively, *strongly* γ-*collectionwise normal*) *in* X if for every discrete collection $\{E_\alpha \mid \alpha < \gamma\}$ of closed subsets of X (respectively, of Y), there is a pairwise disjoint collection $\{U_\alpha \mid \alpha < \gamma\}$ of open subsets of X such that $E_\alpha \cap Y \subset U_\alpha$ (respectively, $E_\alpha \subset U_\alpha$) for every $\alpha < \gamma$ ([18]). Clearly, Y being ω-collectionwise normal (respectively, strongly ω-collectionwise normal) in X is equivalent to that Y is normal (respectively, strongly normal) in X. When Y is γ-collectionwise normal (respectively, strongly γ-collectionwise normal) in X for every γ, we say Y is *collectionwise normal* (respectively, *strongly collectionwise normal) in* X; we see that collectionwise normality (respectively, strongly collectionwise normality) of Y in X is equal to being $\alpha - CN$ (respectively, $\gamma - CN$) of Y in the sense of Aull [7].

2. Preliminaries and 1- or 2- (collectionwise) normality of a subspace in a space

At first, we recall some preliminary notions and facts.

Let Y be a subspace of a space X. As is known, Y is said to be C^*- (respectively, C-) *embedded in* X if every bounded real-valued (respectively, real-valued) continuous function on Y is continuously extended over X. A subspace Y is said to be P^γ- (respectively, P-) embedded in X if every continuous γ-separable (respectively, continuous) pseudo-metric on Y is continuously extended over X ([1]); a pseudo-metric d on Y is γ-*separable* if the pseudo-metric space (Y, d) has weight $\leq \gamma$. It is known that P^ω-embedding is equal to C-embedding ([1]).

By [2], Y is said to be *weakly* C-*embedded* in X if for every real-valued continuous function f on Y there exists a real-valued function on X which is an extension of f and continuous at each point of Y. By [18], Y is said to be *weakly* P^γ- (respectively, *weakly* P-) embedded in X if every continuous γ-separable (respectively, continuous) pseudo-metric on Y is extended to a pseudo-metric on X which is continuous at each point of $Y \times Y$. Weak P^ω-embedding is equal to weak C-embedding ([18]). A space X is γ-*collectionwise normal* if for every discrete collection $\{E_\alpha \mid \alpha < \gamma\}$ of closed subsets there exists a pairwise disjoint collection $\{G_\alpha \mid \alpha < \gamma\}$ of open subsets such that $E_\alpha \subset G_\alpha$ for each $\alpha < \gamma$. Clearly, X is collectionwise normal if X is γ-collectionwise normal for every γ.

A subspace Y is said to be *Hausdorff in* X if for every two distinct points y_1, y_2 of Y, there are disjoint open subsets U_1, U_2 of X such that $y_i \in U_i$ for each $i = 0, 1$. A subspace Y is said to be *strongly regular in* X if for each $x \in X$ and each closed subset F of X with $x \notin F$, there exist disjoint open subsets U, V of X such that $x \in U$ and $F \cap Y \subset V$.

Let X_Y denote the space obtained from the space X, with the topology generated by a subbase $\{U \mid U$ is open in X or $U \subset X \setminus Y\}$. Hence, points in $X \setminus Y$ are isolated and Y is closed in X_Y. Moreover, X and X_Y generate the same topology on Y ([12]). As is seen in [2], the space X_Y is often useful in discussing several relative topological properties. It is easy to see that Y is Hausdorff in X if and only if X_Y is Hausdorff. The following results given in [2], [18] are fundamental in the present paper.

Lemma 2.1 ([2],[18]). *For a subspace Y of a space X the following statements are equivalent.*

(*a*) *Y is strongly normal in X.*

(*b*) *Y is normal in G for every open subset G of X with $Y \subset G$.*

(*c*) *X_Y is normal.*

(*d*) *Y is normal in X_Y.*

(*e*) *Y is normal itself and weakly C-embedded in X.*

Lemma 2.2 ([18]). *For a subspace Y of a space X the following statements are equivalent.*

(*a*) *Y is strongly γ-collectionwise normal in X.*

(*b*) *Y is γ-collectionwise normal in G for every open subset G of X with $Y \subset G$.*

(*c*) *X_Y is γ-collectionwise normal.*

(*d*) *Y is γ-collectionwise normal in X_Y.*

(*e*) *Y is γ-collectionwise normal itself and weakly P^γ-embedded in X.*

Corresponding to Lemmas 2.1 and 2.2 we have the following lemma; $(a) \Leftrightarrow (c)$ was recently obtained in [30], and $(c) \Leftrightarrow (e)$ for Y being Hausdorff in X was proved in [18, Lemma 4.6]. Other equivalences are easily proved.

Lemma 2.3. *For a subspace Y of a space X, the following statements from (a) to (d) are equivalent. If Y is Hausdorff in X, these are equivalent to (e).*

(*a*) *Y is Aull-paracompact in X.*

(*b*) *Y is 2-paracompact in G for every open subset G of X with $Y \subset G$.*

(*c*) *X_Y is paracompact.*

(*d*) *Y is 2-paracompact in X_Y.*

(*e*) *Y is paracompact itself and weakly P-embedded in X.*

We now introduce notions of 1- or 2- (collectionwise) normality of Y in X. We say that a subspace Y of a space X is 1- (respectively, 2-) *normal in X* if for each disjoint closed subsets F_0, F_1 of X there exist open subsets G_0, G_1 of X such that $F_i \cap Y \subset G_i$ for each $i = 0, 1$ and $\{G_0, G_1\}$ is discrete in X (i.e. $\overline{G_0} \cap \overline{G_1} = \emptyset$) (respectively, discrete at each point of Y in X (i.e. $\overline{G_0} \cap \overline{G_1} \cap Y = \emptyset$)).

A subspace Y of a space X is 1-γ- (respectively, 2-γ-) *collectionwise normal in X* if for each discrete collection $\{F_\alpha \mid \alpha < \gamma\}$ of closed subsets of X there exists a collection $\{G_\alpha \mid \alpha < \gamma\}$ of open subsets of X such that $F_\alpha \cap Y \subset G_\alpha$ for each $\alpha < \gamma$ and $\{G_\alpha \mid \alpha < \gamma\}$ is discrete in X (respectively, discrete at each point of

Y in X). If Y is 1- (respectively, 2-) γ-collectionwise normal in X for every γ, Y is said to be 1- (respectively, 2-) *collectionwise normal in* X[†].

In the above definitions of 2-normality and 2-γ-collectionwise normality of Y in X, it is easy to see that both $\{G_1, G_2\}$ and $\{G_\alpha \mid \alpha < \gamma\}$ can be taken to be disjoint. Therefore, 2- (collectionwise) normality of Y in X implies (collectionwise) normality of Y in X.

These definitions above admit the following result; for brevity "cw-normal" means collectionwise normal. Moreover, the symbols "H" and "SR" mean the assumptions that "Y is Hausdorff in X" and "Y is strongly regular in X", respectively.

Proposition 2.4. *For a subspace Y of a space X the following implications hold.*

$$
\begin{tikzcd}
X \textit{ is paracompact} \arrow[d] & X \textit{ is cw-normal} \arrow[d] \arrow[rr] & & X \textit{ is normal} \arrow[d] & \\
\begin{array}{c} Y \textit{ is} \\ 1\textit{-paracompact} \\ \textit{in } X \end{array} \arrow[r, "\mathsf{SR}"] \arrow[d] & \begin{array}{c} Y \textit{ is} \\ 1\textit{-cw-normal} \\ \textit{in } X \end{array} \arrow[rr] \arrow[d] & & \begin{array}{c} Y \textit{ is} \\ 1\textit{-normal} \\ \textit{in } X \end{array} \arrow[d] & \\
\begin{array}{c} Y \textit{ is} \\ 2\textit{-paracompact} \\ \textit{in } X \end{array} \arrow[r, "\mathsf{SR}"] & \begin{array}{c} Y \textit{ is} \\ 2\textit{-cw-normal} \\ \textit{in } X \end{array} \arrow[rr] \arrow[dr] & & \begin{array}{c} Y \textit{ is} \\ 2\textit{-normal} \\ \textit{in } X \end{array} \arrow[dr] & \\
& & \begin{array}{c} Y \textit{ is} \\ \textit{cw-normal} \\ \textit{in } X \end{array} \arrow[rr] & & \begin{array}{c} Y \textit{ is} \\ \textit{normal} \\ \textit{in } X \end{array} \\
\begin{array}{c} Y \textit{ is} \\ \textit{Aull-paracompact} \\ \textit{in } X \end{array} \arrow[r, "\mathsf{H}"] \arrow[uu] \arrow[d, leftrightarrow] & \begin{array}{c} Y \textit{ is} \\ \textit{strongly cw-normal} \\ \textit{in } X \end{array} \arrow[rr] \arrow[uu] \arrow[d, leftrightarrow] & & \begin{array}{c} Y \textit{ is} \\ \textit{strongly normal} \\ \textit{in } X \end{array} \arrow[uu] \arrow[d, leftrightarrow] & \\
X_Y \textit{ is paracompact} \arrow[d] & X_Y \textit{ is cw-normal} \arrow[d] & & X_Y \textit{ is normal} \arrow[d] & \\
Y \textit{ is paracompact} & Y \textit{ is cw-normal} \arrow[rr] & & Y \textit{ is normal} &
\end{tikzcd}
$$

[†]2-collectionwise normality of Y in X is called collectionwise normality of Y in X in a recent paper of E. Grabner, G. Grabner, Miyazaki and Tartir, "*Relative collectionwise normality*" to appear in Appl. Gen. Top. Moreover, they also independently proved the implication "Y is 2-paracompact in $X \xrightarrow{\mathsf{SR}} Y$ is 2-cw-normal in X" in Proposition 2.4 assuming further that X is Hausdorff.

Bella and Yaschenko [8] proved the following theorem. A space X is said to be *almost compact* if for every pair of disjoint zero-sets Z_0, Z_1 in X, either Z_0 or Z_1 is compact. Note that a Tychonoff space X is almost compact if and only if $|\beta X \setminus X| \leq 1$, where βX is the Stone-Čech compactification of X.

Theorem 2.5 ([8]). *For a Tychonoff space Y, the following statemants are equivalent.*

(a) *Y is weakly C-embedded in every larger Tychonoff (or equivalently, regular) space.*

(b) *Y is either Lindelöf or almost compact.*

Theorem 2.5 was improved to the following.

Theorem 2.6 ([18]). *For a Tychonoff space Y, the following statemants are equivalent.*

(a) *Y is weakly P^{γ}-embedded in every larger Tychonoff (or equivalently, regular) space.*

(b) *Y is either Lindelöf or almost compact.*

With Theorem 2.5, Bella and Yaschenko [8] further proved the following theorem, which was independently proved by Matveev et al. [25].

Theorem 2.7 ([8],[25]). *For a Tychonoff (respectively, regular) space Y, the following statemants are equivalent.*

(a) *Y is strongly normal in every larger Tychonoff (respectively, regular) space.*

(b) *Y is normal in every larger Tychonoff (respectively, regular) space.*

(c) *Y is either Lindelöf or normal and almost compact.*

Similarly, Theorem 2.6 and Lemma 2.2 provide the following theorem.

Theorem 2.8 ([18]). *For a Tychonoff (respectively, regular) space Y, the following statemants are equivalent.*

(a) *Y is strongly collectionwise normal in every larger Tychonoff (respectively, regular) space.*

(b) *Y is collectionwise normal in every larger Tychonoff (respectively, regular) space.*

(c) *Y is either Lindelöf or normal and almost compact.*

Remark 2.9. Combining Proposition 2.4 and Theorems 2.7, 2.8, we have that "strongly normal" (respectively, "strongly collectionwise normal") can be replaced by "2-normal" (respectively, "2-collectionwise normal") in Theorem 2.7 (respectively, Theorem 2.8).

Moreover, the following theorem follows from Theorem 2.6 and Lemma 2.3.

Theorem 2.10 ([4], [15], [30]). *For a Tychonoff space Y, the following statemants are equivalent.*

- (a) *Y is Aull-paracompact in every larger Tychonoff (or equivalently, regular) space.*
- (b) *Y is 2-paracompact in every larger Tychonoff (or equivalently, regular) space.*
- (c) *Y is Lindelöf.*

Remark 2.11. In Theorems 2.5, 2.6, 2.7, 2.8 and 2.10, all "larger Tychonoff (respectively, regular) space" can be replaced by "larger Tychonoff (respectively, regular) space containing Y as a closed subspace".

Remark 2.12. Yamazaki [29] showed that the following are equivalent for a Hausdoff space Y:

- (a) Y is weakly C-embedded (or equivalently, weakly P-embedded) in every larger Hausdorff space.
- (b) Y is either compact or every continuous real-valued function on Y is constant.

In the condition (a), "larger Hausdorff space" can be replaced by "larger Hausdorff space containing Y as a closed subspace".

Hence, if we replace all "Tychonoff" in Theorems 2.7, 2.8 and 2.10 by "Hausdorff", the conditions (c) of each theorems are replaced by "Y is compact" (see also [29], [30]).

Remark 2.13. Yamazaki [31] constructed a T_1-space X and a subspace Y such that Y is normal in X, but not 2-normal in X. We do not know similar examples under higher separation axioms. Furthermore, it is unknown whether if 2-normality implies 2-ω-collectionwise normality, or collectionwise normality implies 2-collectionwise normality.

3. Quasi-C*-, quasi-C- and quasi-P$^\gamma$-embeddings

In this section, we introduce new extension properties called quasi-C^*-, quasi-C- and quasi-P-embeddings, which will play basic roles on the study of 1- (collectionwise) normality.

Let X be a space and $\mathcal{E} = \{E_\alpha \,|\, \alpha \in \Omega\}$ a collection of subsets of X. Then $\mathcal{E}$ is said to be *uniformly discrete* in X if there exist a collection $\{Z_\alpha \,|\, \alpha \in \Omega\}$ of zero-sets of X and a discrete collection $\{G_\alpha \,|\, \alpha \in \Omega\}$ of cozero-sets of X such that $E_\alpha \subset Z_\alpha \subset G_\alpha$ for each $\alpha \in \Omega$ ([9]).

Let us now define that a subspace Y of a space X is *quasi-C^*-embedded in X* if for each pair Z_0, Z_1 of disjoint zero-sets of Y, there exist open subsets G_0, G_1 of X such that $\{G_0, G_1\}$ is discrete in X and $Z_i \subset G_i$ for each $i = 0, 1$.

A subspace Y of a space X is said to be *quasi-P^γ-embedded in X* if for each uniformly discrete collection $\{Z_\alpha \mid \alpha < \gamma\}$ of zero-sets of Y, there exists a discrete collection $\{G_\alpha \mid \alpha < \gamma\}$ of open subsets of X such that $Z_\alpha \subset G_\alpha$ for each $\alpha < \gamma$. A subspace Y is *quasi-P-embedded in X* if Y is quasi-P^γ-embedded in X for every γ. Furthermore, quasi-P^ω-embedding is called *quasi-C-embedding.*

Definitions of quasi-C^*-embedding and quasi-P^γ-embedding should be compared with the following results in [9], [18] and [19].

Lemma 3.1 ([9]). *A subspace Y of a space X is P^γ-embedded in X if and only if if for every uniformly discrete collection of subsets of Y of cardinality $\leq \gamma$ is also uniformly discrete in X.*

Lemma 3.2 ([18]). *A subspace Y of a space X is weakly C-embedded in X if and only if if for each pair Z_0, Z_1 of disjoint zero-sets of Y, there exist disjoint open subsets G_0, G_1 of X such that $Z_i \subset G_i$ for each $i = 0, 1$.*

Lemma 3.3 ([19]). *A subspace Y of a space X is weakly P^γ-embedded in X if and only if for each uniformly discrete collection $\{E_\alpha \mid \alpha < \gamma\}$ of zero-sets of Y there exists a pairwise disjoint collection $\{G_\alpha \mid \alpha < \gamma\}$ of open subsets of X such that $E_\alpha \subset G_\alpha$ for each $\alpha < \gamma$.*

By Lemmas 3.1, 3.2 and 3.3, we have the following implications.

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & C^* \arrow[d] \\
\text{quasi-}P \arrow[r] \arrow[d] & \text{quasi-}C \arrow[r] & \text{quasi-}C^* \arrow[d] \\
\text{weak } P \arrow[rr] & & \text{weak } C
\end{tikzcd}$$

We note that none of reverse implications above is true.

Proposition 3.4. *For a subspace Y of a space X, the following statements hold.*

(*a*) *If Y is itself γ-collectionwise normal and quasi-P^γ-embedded in X, then Y is 1-γ-collectionwise normal in X.*

(*b*) *If Y is itself normal and quasi-C^*-embedded in X, then Y is 1-normal in X.*

Moreover, if Y is closed in X, each of (*a*) *and* (*b*) *reverses.*

In [6], Aull defined that a subspace Y of a space X is *α-paracompact in X* if for every collection $\mathcal{U}$ of open subsets of X with $Y \subset \bigcup \mathcal{U}$, there exists a collection $\mathcal{V}$ of open subsets of X such that $Y \subset \bigcup \mathcal{V}$, $\mathcal{V}$ is a partial refinement of $\mathcal{U}$ and $\mathcal{V}$ is locally finite in X. Note that α-paracompactness of Y in X implies Aull-paracompactness of Y in X ([3], [4]).

Related to α-paracompactness, let us recall the following results in [22] and [23, Theorem 1.3].

Theorem 3.5 (**[22]**). *A Hausdorff (respectively, regular, Tychonoff) space Y is α-paracompact in every Hausdorff (respectively, regular, Tychonoff) space containing Y as a closed subspace if and only if Y is compact.*

Theorem 3.6 (**[23]**). *For a closed subspace Y of a regular space X, Y is 1-paracompact in X if and only if Y is α-paracompact in X.*

Theorems 3.5 and 3.6 immediately induce a characterization of absolute 1-paracompactness as follows.

Corollary 3.7. *For a Tychonoff (respectively, regular) space Y, the following statemants are equivalent.*

(*a*) *Y is 1-paracompact in every larger Tychonoff (respectively, regular) space.*
(*b*) *Y is α-paracompact in every larger Tychonoff (respectively, regular) space.*
(*c*) *Y is compact.*

The following is one of our main theorems characterizing absolute quasi-P-, quasi-C- and quasi-C^*-embeddings.

Theorem 3.8. *For a Tychonoff space Y, the following statements are equivalent.*

(*a*) *Y is quasi-P-embedded in every larger Tychonoff space.*
(*b*) *Y is quasi-C-embedded in every larger Tychonoff space.*
(*c*) *Y is quasi-C^*-embedded in every larger Tychonoff space.*
(*d*) *Y is almost compact.*

In the conditions from (a) to (c), "Tychonoff" can be replaced by "regular".

By Proposition 3.4 and Theorem 3.8, we have

Corollary 3.9. *For a Tychonoff (respectively, regular) space Y, the following statements are equivalent.*

(*a*) *Y is 1-collectionwise normal in every larger Tychonoff (respectively, regular) space.*
(*b*) *Y is 1-normal in every larger Tychonoff (respectively, regular) space.*
(*c*) *Y is normal and almost compact.*

In Corollary 3.9, $(b) \Leftrightarrow (c)$ also follows from [25, Theorem 2.6]. For the Hausdorff case, we have the following.

Theorem 3.10. *For a Hausdorff space Y, the following statements are equivalent.*

(a) *Y is quasi-C^*-embedded in every larger Hausdorff space.*

(b) *Every continuous real-valued function on Y is constant.*

In (a), "quasi-C^*-embedded" can be replaced by "quasi-P-embedded" or "quasi-C-embedded" and "larger Hausdorff space" can be replaced by "larger Hausdorff space containing Y as a closed subspace".

By Theorem 3.10 and Proposition 3.4, we have the following; *a Hausdorff space Y is 1-collectionwise normal (or equivalently, 1-normal) in every larger Hausdorff space if and only if $|Y| \leq 1$. Moreover, "larger Hausdorff space" can be replaced by "larger Hausdorff space containing Y as a closed subspace".*

Finally we consider the condition under which 2-paracompactness implies 1-paracompactness. We say a subspace Y of a space X is T_4- (respectively, T_3-) *embedded in X* if for every closed subset F of X disjoint from Y (respectively, $z \in X \setminus Y$), F (respectively, z) and Y are separated by disjoint open subsets of X. The idea of these notions already appeared in Aull [6]. It is easy to see that if Y is T_3-embedded in X, then Y is closed in X.

The following is a finer result of Theorem 3.6; to show "$(b) \Rightarrow (c)$", the implication "$(b) \Rightarrow Y$ is T_4-embedded in X" is due to Aull [6, Theorem 6]. By using this fact, Lupiañez and Outerelo [23, Lemma 1.2 and Theorem1.3] proved $(a) \Rightarrow (c) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$.

Theorem 3.11 ([23]). *For a closed subspace Y of a regular space X the following statements are equivalent.*

(a) *Y is 1-paracompact in X.*

(b) *Y is α-paracompact in X.*

(c) *Y is 2-paracompact in X and T_4-embedded in X.*

The proof of Theorem 3.11 essentially shows the following.

Theorem 3.12. *For a subspace Y of a space X the following statements are equivalent.*

(a) *Y is 1-paracompact in X and T_3-embedded in X.*

(b) *Y is α-paracompact in X and for every $y \in Y$ and every closed subset F of X with $F \cap Y = \emptyset$, there exists an open subset U of X such that $y \in U \subset \overline{U}^X \subset X \setminus F$.*

(c) *Y is 2-paracompact in X and T_4-embedded in X.*

Proposition 3.13. *For a Tychonoff space Y the following statements are equivalent.*

(*a*) *Y is T_4-embedded in every larger Tychonoff space.*

(*b*) *Y is compact.*

Remark 3.14. In Theorem 3.8, Corollaries 3.7 and 3.9, Proposition 3.13, all "larger Tychonoff (respectively, regular) space" can be replaced by "larger Tychonoff (respectively, regular) space containing Y as a closed subspace".

Theorem 2.10, Theorem 3.11 and Proposition 3.13 give an alternative proof to Corollary 3.7.

In case Y is Hausdorff, we have the following; *a Hausdoff space Y is T_4-embedded in every larger Hausdorff space if and only if $Y = \emptyset$.* The similar proof provides the following; *a Hausdorff space Y is 1-paracompact in every larger Hausdorff space if and only if $Y = \emptyset$.* Moreover, in both statements, "larger Hausdorff space" can be replaced by "larger Hausdorff space containing Y as a closed subspace". This should be compared with Theorem 3.5 and Corollary 3.7.

4. On 1-metacompactness of a subspace in a space

In this section, we describe absolute case of 1-metacompactness. A subspace Y of a space X is said to be *1-metacompact in X* if for every open cover $\mathcal{U}$ of X, there exists an open refinement $\mathcal{V}$ of $\mathcal{U}$ such that $\mathcal{V}$ is point-finite at every $y \in Y$ ([21]). In [16], 1-metacompactness of Y in X is called strongly metacompactness of Y in X.

A space X satisfies the *discrete finite chain condition* (DFCC, for short) if every discrete collection of non-empty open subsets of X is finite (see [24], for example). Recall that a Tychonoff space X is pseudocompact if and only if X satisfies the DFCC. It is also known that a Tychonoff space X is compact if and only if X is pseudocompact and metacompact ([27], [28]). Furthermore, a regular space X is compact if and only if X satisfies the DFCC and is metacompact ([27]).

According to [2], in [4], Arhangel'skiĭ and Genedi remarked the following fact; *let Y be a countable dense subset of a regular space X. Then Y is 1-metacompact (or equivalently, 1-paracompact) in X if and only if X is Lindelöf.* The proof of this fact is applied to show the following lemma.

Lemma 4.1. *Take a separable space Z and a non-DFCC space Y, arbitrarily.*

Let $\{d_n \mid n \in \mathbb{N}\}$ be a countable dense subset of Z, $\{U_n \mid n \in \mathbb{N}\}$ a countable discrete collection of non-empty open subsets of Y and $\{y_n \mid n \in \mathbb{N}\}$ a countable *closed discrete subset of Y such that $y_n \in U_n$ for each $n \in \mathbb{N}$. Let X be the quotient space obtained from $Y \oplus Z$ by identifying y_n with d_n for each $n \in \mathbb{N}$.*

If Y is 1-metacompact in X, then Z is Lindelöf.

Moreover, if Y and Z are Tychonoff (respectively, regular), then X is also Tychonoff (respectively, regular).

Theorem 4.2. *A Tychonoff (respectively, regular, Hausdorff) space Y is 1-metacompact in every larger Tychonoff (respectively, regular, Hausdorff) space if and only if Y is compact.*

Theorem 4.2 extends the following result due to E. Grabner et al. [16]; *a normal space Y is 1-metacompact in every larger regular space if and only if Y is compact.*

5. On 1-subparacompactness of a subspace in a space

It was defined in [26] that a subspace Y of a space X is *1-subparacompact in* X if for every open cover $\mathcal{U}$ of X, there exists a σ-discrete collection $\mathcal{P}$ of closed subsets of X with $Y \subset \bigcup \mathcal{P}$ such that $\mathcal{P}$ is a partial refinement of $\mathcal{U}$.

In [26], Qu and Yasui asked a question as follows; *let X be a regular space and Y a subspace of X. Is it true that if Y is 1-paracompact in X, then Y is 1-subparacompact in X?* The following theorem gives a negative answer to this question.

Theorem 5.1. *There exists a Tychonoff space X and a subspace Y of X such that Y is 1-paracompact but not 1-subparacompact in X.*

Construction. Let X be the set $(\omega_2 + 1) \times (\omega_1 + 1) \setminus \{\langle \omega_2, \omega_1 \rangle\}$. For $\alpha \in \omega_1$ and $\beta \in \omega_2$, define $G_\alpha = (\omega_2 + 1) \times \{\alpha\}$ and $H_\beta = \{\beta\} \times (\omega_1 + 1)$, respectively. Define a topology on X as follows. For $\alpha \in \omega_1$, a neighborhood base at $\langle \omega_2, \alpha \rangle$ is the family of all sets of the form $G_\alpha \setminus E$, where E is a finite subset of $\omega_2 \times \{\alpha\}$. For $\beta \in \omega_2$, a neighborhood base at $\langle \beta, \omega_1 \rangle$ is the family of all sets of the form $H_\beta \setminus F$, where F is a finite subset of $\{\beta\} \times \omega_1$. All other points of X are isolated in X. The construction of X is based on a example in [11]. Let $Y = X \setminus \big((\omega_2 \times \{\omega_1\}) \cup (\{\omega_2\} \times \omega_1)\big)$. Then Y is 1-paracompact but not 1-subparacompact in X.

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